Problem 1.

(a) The jet moves with respect to the fixed lab frame with Lorentz factor $\Gamma = (1-\beta^2)^{-1/2}$. A photon has energy E and angle θ in the rest frame of the jet (θ is measured with respect to the radial direction). Find the photon energy in the lab frame, $E_{\rm lab}$. The ratio $E_{\rm lab}/E$ defines the Doppler factor,

$$\mathcal{D} = \frac{E_{\rm lab}}{E}. \tag{1}$$

In essence, you are asked to derive $\mathcal{D}$ in terms of Γ and θ. What are the values of $\mathcal{D}$ for $\theta = 0, \pi/2, \pi$?

[Hint: use Lorentz transformation of the photon 4-momentum $p^\alpha = (p^0, p^1, p^2, p^3)$ where $p^0 = E/c$.]

(b) Show that the photon angle in the lab frame, $\theta_{\rm lab}$, is related to its angle in the jet frame, θ, by

$$\sin\theta_{\rm lab} = \mathcal{D}^{-1}\sin\theta. \tag{2}$$

[Hint: $\cos\theta = p^1/p^0$ where p^1 is the radial component of the photon momentum. A similar expression defines $\cos\theta_{\rm lab}$ in the lab frame, so you can find the transformation of $\cos\theta$ using the Lorentz transformation of p^α. Once you obtain the expression for $\cos\theta_{\rm lab}$ in terms of $\cos\theta$, consider $\sin^2\theta_{\rm lab} = 1 - \cos^2\theta_{\rm lab}$.]

(c) Consider a photon propagating between two successive scatterings in a jet whose Lorentz factor $\Gamma \gg 1$ increases linearly with radius. Show that the photon propagation occurs with $\theta = const$, where θ is photon angle measured in the rest frame of the jet.

[Hint: use the relations in (a) and (b). Use the condition $\Gamma \gg 1$ by taking $\beta \approx 1$ in the expression for the Doppler factor.]

Problem 2.

Consider a hot relativistic outflow with luminosity L and energy per baryon $\eta c^2 = L/\dot{M} \gg 1$. The flow starts at radius R_0 with a small Lorentz factor $\Gamma_0 \sim (1-\beta_0)^{-1/2} \sim 1$ and mildly relativistic β_0 ($\Gamma_0\beta_0 \approx 1$). Suppose that the flow is collimated along the z axis so that its perpendicular cross section $S(r)$ is given by

$$S(r) = S_0\left(\frac{r}{R_0}\right)^{\psi}. \tag{3}$$

A spherically expanding flow (no collimation) would have $S(r) = 4\pi r^2$ (i.e. $\psi = 2$); its adiabatic cooling and acceleration was described in the lecture. Your task is to derive similar solutions for the more general case where ψ may differ from 2.

(a) Show that $T(r) = T_0/\Gamma(r)$ and find $\Gamma(r)$ at small radii where the flow energy is dominated by radiation.

(b) For the spherical flow, the photon-to-baryon ratio n_γ/n remains constant (see lecture). Does this result hold for collimated flows with $\psi \neq 2$?

(c) Evaluate the radius $R_{\rm sat}$ at which the jet Lorentz factor saturates near its maximum. What is the maximum/saturation value of Γ?

(d) Evaluate the photospheric radius of the outflow $R_{\rm ph}$ assuming $R_{\rm ph} > R_{\rm sat}$.

(e) For fixed L and R_0, find the range of η for which $R_{\rm ph} > R_{\rm sat}$ (matter-dominated photosphere).

Problem 3.

Consider a spherically expanding outflow at radii $R_{\rm sat} < r < R_{\rm ph}$. The proton component of the jet is hot, $kT_p \sim m_p c^2$. Protons heat the electrons via Coulomb collisions. The outflow power L and energy per baryon η are given.

(a) Find the net energy given by protons to the electrons at radii $R_{\rm ph}/30 < r < R_{\rm ph}$ (the region of optical depth $30 < \tau_{\rm T} < 1$).

(b) Estimate the electron temperature at which Compton cooling balances Coulomb heating. You can assume that the parameter $\epsilon = U_{\rm rad}/\rho c^2$ is given, where $U_{\rm rad}$ is the energy density of radiation in the rest-frame of the flow, and ρ is the proper mass density of the flow (ϵ is basically the fraction of the flow energy that is carried by radiation, expressed in the rest frame where matter is at rest.)

Central object:

size	$R_0 \sim 10^6 - 10^7 \text{cm}$
activity duration	$t_{burst} \sim 0.1 - 100 \text{ s}$
Power (isotropic equivalent)	$L \sim 10^{51} - 10^{53} \text{erg/s}$

Energy density at the base of the jet:

$$U_0 \sim \frac{L}{R_0^2 c} \sim 10^{27} L_{52} R_{0,7} \frac{\text{erg}}{\text{cm}^3} \qquad \left(\frac{U_0}{c^2} \sim 10^6 \text{g cm}^{-3} \right)$$

Relativistic explosion: $U_0 >> \rho_0 c^2$

Energy density dominated by radiation (and $e^{\pm}$ pairs):

$$a T_0^4 \sim U_0 \quad \Rightarrow \quad T_0 \sim 10^{10} K, \qquad kT_0 \sim 1 \text{ MeV}$$

Relativistic fluid description

proper baryon density	$n, \quad \rho = mn$
four-velocity	$u^\alpha = (\Gamma, \Gamma\vec{\beta})$ (local orthonormal basis)
stress-energy tensor	$T^{\alpha\beta} = w\, u^\alpha u^\beta + g^{\alpha\beta} P$
enthalpy	$w = \rho c^2 + U + P$

Steady spherical outflow

Baryon number conservation: $\dot{M} = 4\pi r^2 \rho\, u^1 c = const$

$\left[\nabla_\alpha (\rho\, u^\alpha) = 0\right]$ $\qquad u^1 = \Gamma\beta$

Energy conservation: $L = 4\pi r^2\; T^{01} c = const$

$\left[\nabla_\alpha T^{\alpha\beta} = 0\right]$ $\qquad T^{01} = w\, \Gamma^2 \beta$

$$L = 4\pi r^2 w\, \Gamma^2 \beta c = const, \qquad \dot{M} = 4\pi r^2 \rho\, \Gamma \beta c = const \quad (1)$$

energy per baryon $\boxed{\eta\, c^2 = \frac{L}{\dot{M}} = \frac{w\Gamma}{\rho}} = const$ (2)

Radiation-dominated: $w = U + P, \quad P = \frac{U}{3}$

Adiabatic cooling: $\mathrm{P} \propto \rho^{4/3} \Rightarrow \boxed{w \propto \rho^{4/3}}$ (3)

(2), (3) $\Rightarrow \rho \propto \Gamma^{-3}, \quad w = \frac{4}{3} a T^4 \propto \Gamma^{-4}$

$\Downarrow \qquad \boxed{\Gamma T = \Gamma_0 T_0 \approx T_0}$

(1) $\Rightarrow \boxed{\Gamma \approx r / R_0}$

End of acceleration where $w \sim \rho c^2$ or $\Gamma \approx \eta$ (eq. 2)

Γ approaches η and saturates at radius $\boxed{R_{\rm sat} = \eta R_0}$

Photon number density $n_\gamma = \frac{aT^4}{2.7kT} \propto T^3 \propto \Gamma^{-3}$

Baryon number density $n = \frac{\rho}{m_p} \propto \Gamma^{-3}$

$\Rightarrow$ photon-to-baryon ratio $\frac{n_\gamma}{n_b} = const$

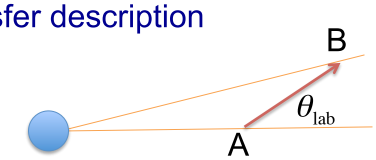
$\Rightarrow$ photon number is conserved

$\Rightarrow$ energy per photon measured in the fixed lab frame $\bar{E} = const$

(also seen from $\bar{E} \propto \Gamma T = const$)

$$\frac{n_\gamma}{n_b} \approx 240\ \eta \left(\frac{r_0}{10^7\,\mathrm{cm}}\right)^{-1/2} \left(\frac{L}{10^{52}\,\mathrm{erg\ s^{-1}}}\right)^{1/4}$$

Radiative transfer description



$$\sin\theta = D\sin\theta_{\rm lab}$$
$$D = \Gamma(1+\cos\theta)$$
$$(\Gamma >> 1, \ \ \beta \approx 1)$$

Exercise:

$\Gamma \propto r \ \Rightarrow$ photon angle θ measured in the outflow rest frame does not change during free propagation between scattering events A and B, θ=const.

- freely propagating radiation remains isotropic in the outflow rest frame
- scattering has no effect on the radiation field and mimics free propagation in vacuum

$\Rightarrow$ Short variability timescale is possible, $\sim R_0/c$?

Optical depth

Optical depth along the ray: $\tau = \int \sigma n_e \, ds$

Consider a photon emitted along the radial direction at a radius r:

$$\tau(r) = \int_r^\infty \sigma_{\rm lab} n_{\rm lab} \, dr'$$

$$\sigma_{\rm lab} = (1-\beta)\ \sigma_{\rm T} \approx \frac{\sigma_{\rm T}}{2\Gamma^2} \quad (\Gamma >> 1)$$

$$n_{\rm lab} = \frac{\dot{M}}{4\pi r^2 m_p \ c}$$

$$\tau(r) = \frac{\dot{M}\sigma_{\rm T}}{8\pi \ m_p c}\int_r^\infty \frac{dr'}{\Gamma'^2 r'^2} = \frac{\dot{M}\sigma_{\rm T}}{8\pi \ m_p c \ \Gamma^2 r} \quad \text{if } \ \Gamma \approx const \quad (r > R_{\rm sat})$$

$$\times \ 1/3 \quad \text{if } \ \Gamma' \propto r' \quad (r << R_{\rm sat})$$

Photosphere: $$\tau \sim \frac{\dot{M}\sigma_{\rm T}}{8\pi \ m_p c \ \Gamma^2 r} \sim 1$$

$$R_{\rm ph} \approx \left(\frac{\dot{M}\sigma_{\rm T}}{8\pi\ m_p c\ R_0} \right)^{1/3} R_0$$

$\left(R_{\rm ph} < R_{\rm sat}\ \text{radiation-dominated photosphere} \right)$

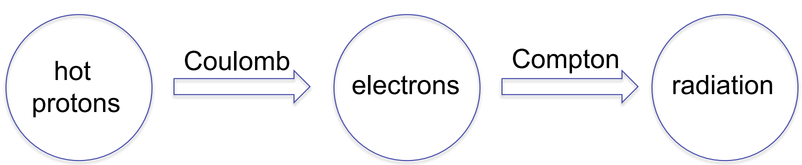
$$R_{\rm ph} \approx \frac{\dot{M}\sigma_{\rm T}}{8\pi\ m_p c\ \eta^2}$$

$\left(R_{\rm ph} > R_{\rm sat}\ \text{matter-dominated photosphere} \right)$

Hot protons give energy to electrons via Coulomb collisions.

(proton loses energy with rate $\dot{E}_{\rm Coul} \approx \frac{3}{2} \ln\Lambda\ \sigma_{\rm T} n_e m_{\rm e} c^3$)

Electrons immediately (on the Compton timescale $t_{\rm C} << t_{\rm exp}$) pass the received energy to radiation.



$$kT_p \sim m_p c^2 \quad >> \quad kT_e \quad > \quad kT_{\rm rad} \sim 1\ \text{keV}$$