Problem set for lectures on radiative processes

1.–Proton synchrotron radiation. The relativistic shock wave propagating with Lorentz factor $\Gamma = 100$ accelerates electrons and protons. The distribution of protons injected at a shock $Q_{\rm p}(\gamma_{\rm p}) \propto \gamma_{\rm p}^{-\Gamma_{\rm inj}}$ extends from $\gamma_{\rm p,min} = \Gamma$ to $\gamma_{\rm p,max} = 10^6$ with a power-law of index $\Gamma_{\rm inj} = 2.5$. The electrons are injected with a power-law of the same index and $\gamma_{\rm e,min} = \frac{1}{3}\Gamma m_{\rm p}/m_{\rm e}$ to $\gamma_{\rm e,max} = 10^9$.
(a) Estimate the cooling timescale of electrons and protons by synchrotron radiation. (Assume magnetic field behind the shock $B = 1$ G.)
(b) Compare the cooling timescales of an electron and a proton of the same energy. Compare the cooling timescales of an electron and a proton of the same Lorentz factor. (Do not forget that the Thomson cross-section is $\sigma_{\rm T} \propto e^4/m^2$ is different for electron and proton.) Compare the cooling timescales to the typical dynamical (escape) timescale $t_{\rm esc} = R/c$, taking $R = 10^{16}$ cm. What the proton energy has to be for it to have cooling timescale comparable to the escape time?
(c) How large fraction of energy a typical proton and a typical electron lose on timescale $t_{\rm esc}$?
(d) Compute the cyclotron frequency for both electrons and protons. At what energies the synchrotron νF_ν spectra from electrons and protons peak? What are the maximum energies the synchrotron spectra extend? What is the steady-state spectrum of electrons and protons?

2.–Compton scattering in Klein-Nishina regime. Consider injection of relativistic electrons into the isotropic radiation field of broad-line region near a quasar. Assume that the radiation field is mono-energetic and consists only of Lyα photons (10.2 eV) of total energy density $U_{rad} = 1$ erg cm^{-3}. The electrons are injected with a power-law distribution extending from $\gamma_{\rm min} = 10$ to $\gamma_{\rm max} = 10^7$ at a rate $Q(\gamma) \propto \gamma^{-\Gamma_{\rm inj}}$ with $\Gamma_{\rm inj} = 2$.
(a) Compute the cooling timescale for Compton scattering for a few representative electron Lorentz factors. Find the steady state electron distribution assuming that the escape time $t_{\rm esc} = R/c$ is much longer than the cooling time-scale. Account for the Klein-Nishina effect using approximate formulae from Moderski et al. (2005). Find the asymptotic power-law indices in the Thomson regime and the Klein-Nishina regime at high-energies. At what electron $\gamma_{\rm br}$ do you expect a break in the electron distribution?
(b) Assume now that these electrons also produce synchrotron radiation. Compute its spectrum and find the photon power-law indices at low and high energies and the difference between them.

3.–Pair production in blazar 3C 454.3. *Fermi Large Area Telescope* has observed a spectral break in the spectrum of this blazar (at redshift $z = 0.859$) at $\sim$2.5 GeV. Detailed modeling of the *Fermi* spectrum above 20 GeV also shows that the maximum optical depth by photon-photon ($\gamma\gamma$) absorption due to the Lyα photons (produced in the BLR) is $\tau_{\gamma\gamma,\rm max} \approx 5$. The observations by the *Hubble Space Telescope* allowed to estimate the Lyα luminosity in this object of $L = 10^{45}$ erg s^{-1}.
(a) Compute the threshold energies for $\gamma\gamma$-absorption for interaction with Lyα and Ly recombination continua of different hydrogen-like atoms (with only one e^- left): hydrogen, He, C, O. [Hint: the ionization energy scales with atomic charge as Z^2.]
(b) Associating the $\sim$2.5 GeV break with absorption of the incident radiation in the broad-line region (BLR), find the energy of the soft photons absorbing the GeV radiation. What atom and transition this energy might correspond to?
(c) Estimate the size R of the BLR where Lyα photons are produced. Assume that BLR photons are distributed homogeneously through out the region of size R, they are isotropic, and the γ-rays are produced at distances much smaller than R. Hint: to estimate the optical depth, first find the number density of Lyα photons.

Compton scattering and synchrotron cooling

Consider an electron of energy $\gamma m_e c^2$ in a radiation field of energy density U_{rad} [erg cm^{-3}]. The radiated power (for $\gamma \gg 1$, for $h\nu\gamma \ll m_e c^2$)

$$\langle P_{Compton} \rangle = \frac{4}{3}\beta^2\gamma^2 c\sigma_T U_{rad} \quad \text{erg s}^{-1}$$

Compare to synchrotron radiation where an electron moves in a magnetic field (consisting of virtual photons) with energy density U_B. For synchrotron radiation we had

$$\langle P_{synchro} \rangle = \frac{4}{3}\beta^2\gamma^2 c\sigma_T U_B \quad \text{erg s}^{-1}$$

where the mean is over the solid angle. The expressions are fully identical, although the processes seem so apparently different.

$\sigma_T = \frac{8\pi}{3} r_e^2 = \frac{2}{3} 10^{-24}$ cm^2 - Thomson cross-section

$r_e = e^2/mc^2$ - classical electron radius

Synchrotron radiation

Cooling Time or Radiative Lifetime

Consider how the electron loose energy. The energy equation becomes:

$$mc^2\frac{d\gamma}{dt} = -P_{emitted} = -2\sigma_T c(\gamma^2\beta^2)U_B \sin^2\alpha \qquad (2)$$

The typical timescale for the electron to loose about half of its energy (i.e. cooling time) is approximately

$$\begin{aligned} t_{cool} &= \frac{Energy}{cooling\ rate} = \frac{\gamma mc^2}{-mc^2\frac{d\gamma}{dt}} = \frac{\gamma mc^2}{P_{emitted}} \\ &= \frac{4\pi mc^2}{\sigma_T c}\frac{1}{\gamma B^2 \sin^2\alpha} = \frac{15\ years}{\gamma B^2 \sin^2\alpha} \qquad (3) \end{aligned}$$

thus for $\gamma = 10^3$ this results in the following cooling times:

Synchrotron radiation: spectrum

Frequency Distribution for a Power-law Electron Distribution

Cosmic rays (protons and ions) that hit Earth have a power-law energy distribution. Reasonable to expect that relativistic electrons also have power-law distributions due to acceleration processes in the Universe. The spectrum from an electron distribution with $n(\gamma)d\gamma$ cm^{-3} electrons between $\gamma m_e c^2$ and $(\gamma + d\gamma)m_e c^2$ is given by

$$j_\nu = \int_1^\infty P_\nu(\gamma) n(\gamma) d\gamma \quad \text{erg/s/Hz/cm}^3$$

A power-law distribution is

$$n(\gamma)d\gamma = n_0 \gamma^{-\Gamma} d\gamma, \quad \gamma_{\min} < \gamma < \gamma_{\max}$$

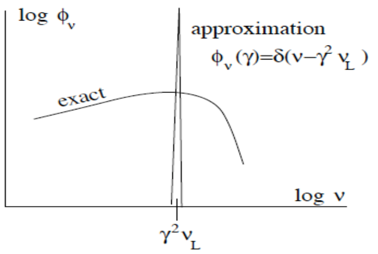
where typically $\Gamma = 2 - 3$.

Synchrotron radiation (powerlaw electrons)

Hard to do analytical integration using exact $P_\nu(\gamma)$. Simpler to approximate $\phi_\nu(\gamma)$, e.g. by assuming that all emission occurs at $\nu \approx \gamma^2 \nu_L$, i.e. $\phi_\nu(\gamma) = \delta(\nu - \gamma^2 \nu_L)$.

Cyclotron frequency

$$\nu_L = \frac{eB}{2\pi mc}$$



$$\begin{aligned} j_\nu &= \frac{4}{3} c\sigma_T U_B n_0 \int_{\gamma_{\min}}^{\gamma_{\max}} \frac{d\gamma}{\gamma^\Gamma} \beta^2 \gamma^2 \delta(\nu - \gamma^2 \nu_L) = \\ &= \frac{2}{3} c\sigma_T U_B n_0 \frac{1}{\nu_L} (\gamma^{1-\Gamma})_{\gamma=\sqrt{\nu/\nu_L}} = \frac{2}{3} c\sigma_T U_B \frac{n_0}{\nu_L} \left(\frac{\nu}{\nu_L}\right)^{-\frac{\Gamma-1}{2}} \quad (5) \end{aligned}$$

where $\gamma_{\min}^2 \nu_L < \nu < \gamma_{\max}^2 \nu_L$. The exponent $\frac{\Gamma-1}{2}$ is called the spectral index.

Compton scattering

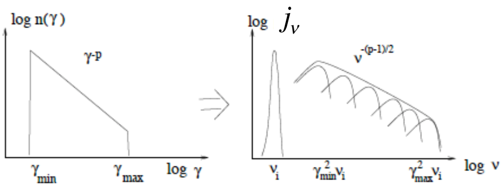
Spectrum from a power-law distribution of relativistic electrons

As for synchrotron case, we use the δ-function approximation

$$\phi_\nu(\gamma) \approx \delta(\nu - \gamma^2 \nu_i)$$

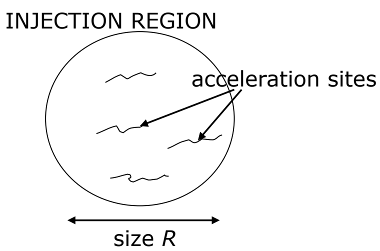
A power-law electron distribution, $n(\gamma)d\gamma = n_0 \gamma^{-p} d\gamma$ then scatters photons of frequency ν_i into a power-law

$$j_\nu = \frac{2}{3} c \sigma_T U_{rad} \frac{n_0}{\nu_i} \left(\frac{\nu}{\nu_i} \right)^{-(p-1)/2} \qquad (7)$$



Compton scattering by relativistic electrons with a power-law distribution gives rise to a power-law distribution of photons.

Kinetic equation for electrons

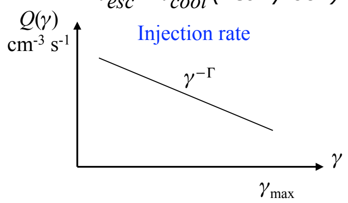


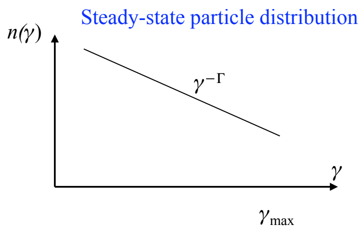
Injection is homogeneous.
The space inbetween acceleration sites is filled with accelerated electrons trying to escape the injection region on a time-scale

$t_{esc} = (R/c)$ x diffusion factor

Before escaping they may cool on a time-scale t_{cool}

1. $t_{esc} < t_{cool}$ *(Leaky box)*





When cooling is inefficient compared to escape, then the steady-state density distribution is $n(\gamma) = t_{esc}\, Q(\gamma)$, i.e. the same shape.

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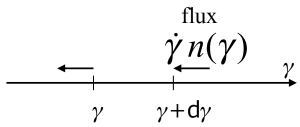
Kinetic equation for electrons

2. $t_{esc} > t_{cool}$ particles cool before escaping

$$\frac{\partial n(\gamma)}{\partial t} + \frac{\partial}{\partial \gamma}[\dot{\gamma}\, n(\gamma)] = Q(\gamma)$$

energy loss rate=
velocity in γ-space



Steady-state particle distribution

$$n(\gamma) = \frac{\int_{\gamma}^{\gamma_{max}} Q(\gamma')d\gamma'}{-\dot{\gamma}(\gamma)}$$

Examples:

i) monoenergetic injection

$$Q(\gamma) = \delta(\gamma - \gamma_{max}) \Rightarrow n(\gamma) \propto \frac{1}{\dot{\gamma}(\gamma)}$$

ii) power-law injection

$$Q(\gamma) = \gamma^{-\Gamma} \text{ if } \Gamma < 1 \Rightarrow n(\gamma) \propto \frac{1}{\dot{\gamma}(\gamma)}$$

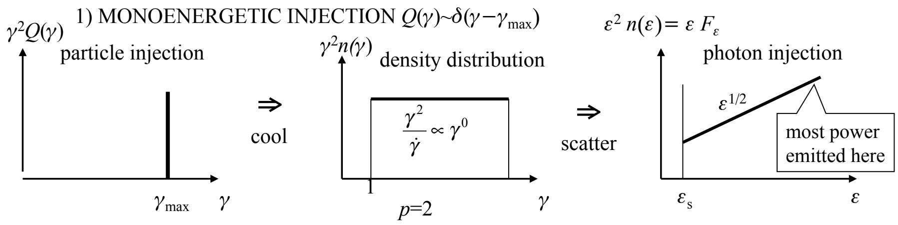
$$\text{if } \Gamma > 1 \Rightarrow n(\gamma) \propto \frac{1}{\gamma^{\Gamma-1}\dot{\gamma}(\gamma)}$$

For synchrotron (and Compton) cooling $\dot{\gamma} \propto \gamma^2$ thus $n(\gamma) \propto 1/\gamma^2$ (for $\Gamma < 1$)

$n(\gamma) \propto 1/\gamma^{\Gamma+1}$ (for $\Gamma > 1$)

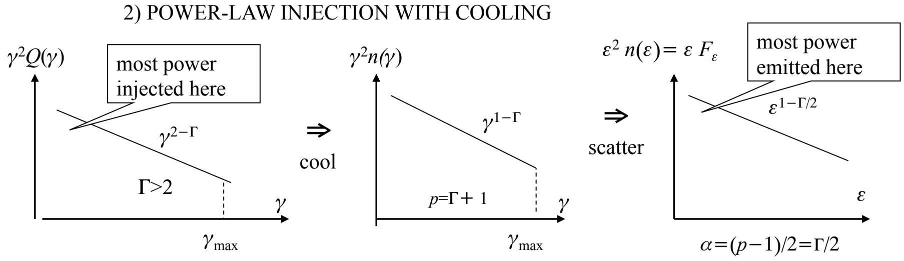
Kinetic equation for electrons

1) MONOENERGETIC INJECTION $Q(\gamma) \sim \delta(\gamma - \gamma_{max})$



Define spectral index $F_\varepsilon \sim \varepsilon^{-\alpha}$; $\alpha = (p-1)/2$

2) POWER-LAW INJECTION WITH COOLING



Shock acceleration ⇒ Γ>2 ⇒ most power at low energies α>1

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Compton scattering: Klein-Nishina effect

If in the electron rest frame the photon energy exceeds electron rest mass, the cross-section for interaction decreases dramatically.

$$|\dot{\gamma}|_{\rm IC} = \frac{4c\sigma_{\rm T}}{3m_{\rm e}c^2} u_0 \gamma^2 F_{\rm KN} \quad \text{where} \quad F_{\rm KN} = \frac{1}{u_0}\int_{\epsilon_{0,\min}}^{\epsilon_{0,\max}} f_{\rm KN}(\tilde{b}) u_{\epsilon_0}\, {\rm d}\epsilon_0$$

$$f_{\rm KN} \simeq \frac{1}{(1+\tilde{b})^{1.5}} \qquad \tilde{b} = 4\gamma x_0, \quad x_0 = h\nu_0 / m_e c^2$$

Moderski et al. 2005, MNRAS, 363, 954

$u_0 = \int_{\epsilon_{0,\min}}^{\epsilon_{0,\max}} u_{\epsilon_0}\, {\rm d}\epsilon_0$ is the total energy density of the radiation field

Compton scattering: Klein-Nishina effect

The kinetic equation for electrons is complicated:

$$\frac{\partial n_\gamma}{\partial t} = -\frac{\partial}{\partial \gamma}(n_\gamma |\dot{\gamma}|) - n_\gamma \int_1^{\gamma} C(\gamma,\gamma')\,{\rm d}\gamma' + \int_{\gamma}^{\gamma_{\max}} n_\gamma C(\gamma',\gamma)\,{\rm d}\gamma' + Q,$$

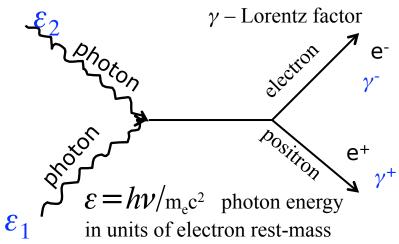
$|\dot{\gamma}|$ is the energy loss rate due to synchrotron radiation
The transition rates $C(\gamma, \gamma')$ have been derived
by Jones (1968) for a mono-energetic ambient radiation field

But approximately kinetic equation can be written as

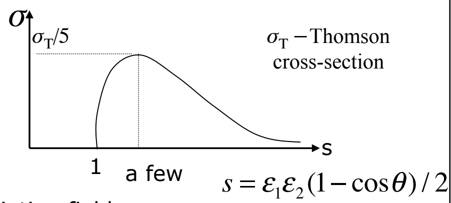
$$\frac{\partial n_\gamma}{\partial t} = -\frac{\partial}{\partial \gamma}(n_\gamma |\dot{\gamma}|_{\rm tot}) + Q$$

where $|\dot{\gamma}_{\rm tot}| = |\dot{\gamma}_{\rm s}| + |\dot{\gamma}_{\rm IC}|$ is cooling rate by synchrotron and Compton accounting for the KN correction.

Photon-photon pair production

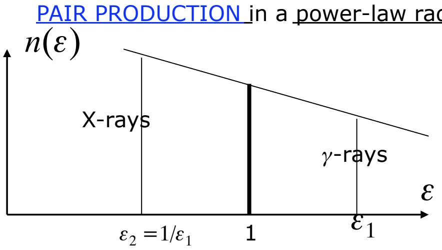


Cross-section for pair production



$s = \varepsilon_1 \varepsilon_2 (1 - \cos\theta)/2$

PAIR PRODUCTION in a power-law radiation field



Produced pairs
Energy conservation
$\gamma^+ + \gamma^- = \varepsilon_2 + \varepsilon_1 \approx \varepsilon_1$
As $\gamma^+ \approx \gamma^- \Rightarrow \gamma^+ \approx \gamma^- \approx \varepsilon_1/2$
Each particle has approximately half of the hard photon energy.

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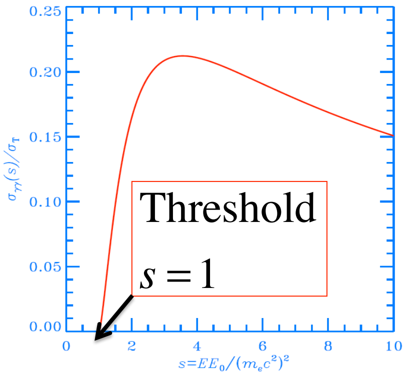
Photon at ε_1 interact mainly with target photons just above threshold at $\varepsilon_2 = 1/\varepsilon_1$

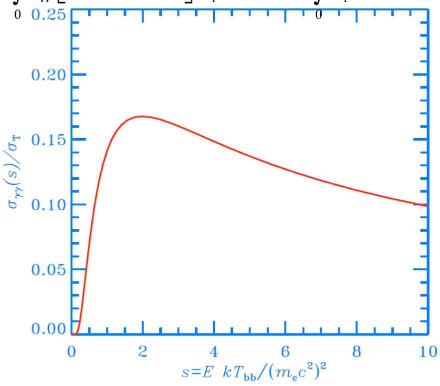
Photon-photon pair production

Angle-averaged cross-section for line target photons
(Gould & Schreder 1967, Zdziarski 1988, Aharonian 2004)

Angle-averaged cross-section for black body target photons

$$\bar{\sigma}_{\gamma\gamma}(E) = \int_0^\infty \sigma_{\gamma\gamma}\left[EE_0/(m_e c^2)^2\right] N_{ph}(E_0)\, dE_0 \Big/ \int_0^\infty N_{ph}(E_0)\, dE_0$$





Optical depth through the photon field of size R

$$\tau_{\gamma\gamma}(E) = \sigma_{\gamma\gamma}(s)\, R\, n_{ph}(E_0) = \sigma_T\, R\, n_{ph}(E_0) \frac{\sigma_{\gamma\gamma}(s)}{\sigma_T}$$

n_{ph} – photon density at energy E_0

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