Summer school "high energy astrophysics", Aug. 22-26, 2011, Weesenstein
Problem set "photohadronic processes and neutrinos" **Walter Winter**

Problem 1: Photohadronic interactions

Interaction rate. The interaction rate of initial nucleons p with photons γ for an interaction type "IT" is given by

$$\Gamma^{\mathrm{IT}}_{p\,\gamma\to p'\,b}(E_p) = \int d\varepsilon \int \frac{d\cos\theta_{p\gamma}}{2} \left(1-\cos\theta_{p\gamma}\right) n_\gamma(\varepsilon,\cos\theta_{p\gamma})\, \sigma^{\mathrm{IT}}(\epsilon_r)\,. \tag{1}$$

Here $n_\gamma(\varepsilon,\cos\theta_{p\gamma})$ is the photon density as a function of photon energy ε and the angle between the photon and proton momenta $\theta_{p\gamma}$ ($\theta_{p\gamma}=\pi$ for heads-on collisions), $\sigma^{\mathrm{IT}}(\epsilon_r)$ is the photon production cross section, and

$$\epsilon_r = \frac{E_p\varepsilon}{m_p}(1-\cos\theta_{p\gamma}) \tag{2}$$

is the photon energy in the nucleon/parent rest frame (PRF). The interaction itself, and therefore E_p and ε, is described in the shock rest frame (SRF). The daughter particles b are often pions.

a) Eliminate the $\cos\theta_{p\gamma}$-dependence from the interaction rate by re-writing the second integral in Eq. (1) as one over ϵ_r for the following cases:

1. Isotropically distributed target photons, *i.e.*, $n_\gamma(\varepsilon,\cos\theta_{p\gamma}) = n_\gamma(\varepsilon)$.
   What does the integral over ϵ_r mean?
2. Heads-on collisions, *i.e.*, $n_\gamma(\varepsilon,\cos\theta_{p\gamma}) = n_\gamma(\varepsilon)\,\delta(\cos\theta_{p\gamma}+1)$.
   In which astrophysical environments may one expect this case?

Hint: The result in 1. is given by

$$\Gamma^{\mathrm{IT}}_{p\,\gamma\to p'\,b}(E_p) = \frac{1}{2}\frac{m_p^2}{E_p^2} \int\limits_{\frac{\epsilon_{\mathrm{th}} m_p}{2E_p}}^{\infty} d\varepsilon \frac{n_\gamma(\varepsilon)}{\varepsilon^2} \int\limits_{\epsilon_{\mathrm{th}}}^{2E_p\varepsilon/m_p} d\epsilon_r\, \epsilon_r\, \sigma^{\mathrm{IT}}(\epsilon_r)\,. \tag{3}$$

Where do the integration limits come from?

Pion production. The pion injection rate is given by

$$Q_\pi(E_\pi) = \sum_{\mathrm{IT}} \int dE_p\, N_p(E_p)\, \Gamma^{\mathrm{IT}}_{p\to\pi}(E_p)\, \frac{dn^{\mathrm{IT}}_{p\to\pi}}{dE_\pi}(E_p,E_\pi)\,. \tag{4}$$

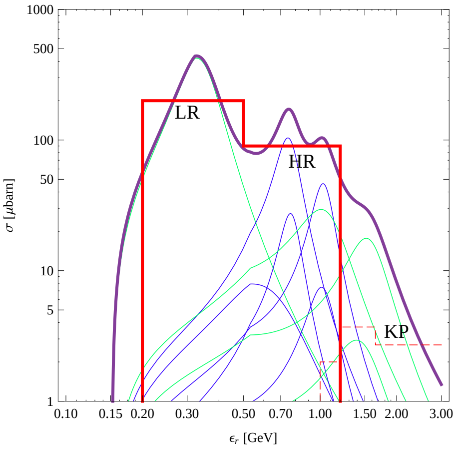

Figure 1: Cross section for the resonances as a function of ϵ_r (thick curve). Green (light gray) resonances are Δ-resonances, blue (dark gray resonances) are N-resonances. Taken from *Hümmer, Rüger, Spanier, Winter, 2010.*

b) Compute the pion injection rate using the simplified scaling function

$$\frac{dn^{\mathrm{IT}}_{p\to\pi}}{dE_\pi}(E_p, E_\pi) \simeq \delta(E_\pi - \chi^{\mathrm{IT}}\, E_p) \cdot M^{\mathrm{IT}}_\pi\,, \tag{5}$$

which assumes that the pions take a fraction of χ^{IT} of the proton energy, and that M^{IT}_π is the pion multiplicity (depending on the pion species). Use the interaction rate from Eq. (3) (result from a-1) and express result in terms of a double integral over $N_p(E_p)$, $n_\gamma(\varepsilon)$, and a "response function" $R^{\mathrm{IT}}_\pi(x, y)$ with

$$x \equiv \frac{E_\pi}{E_p} \quad \text{and} \quad y \equiv \frac{E_p\,\varepsilon}{m_p}\,. \tag{6}$$

What is the meaning of the new variable y?

Hint: The result is given by

$$Q_\pi(E_\pi) = \int\limits_{E_\pi}^{\infty} \frac{dE_p}{E_p}\, N_p(E_p) \int\limits_{\frac{\epsilon_{\mathrm{th}} m_p}{2E_p}}^{\infty} d\varepsilon\, n_\gamma(\varepsilon)\, R^{\mathrm{IT}}_\pi(x, y) \tag{7}$$

with

$$R^{\mathrm{IT}}_\pi(x, y) = \delta\left(x - \chi^{\mathrm{IT}}\right)\, M^{\mathrm{IT}}_\pi\, f^{\mathrm{IT}}(y)\,, \quad f^{\mathrm{IT}}(y) \equiv \frac{1}{2y^2} \int\limits_{\epsilon_{\mathrm{th}}}^{2y} d\epsilon_r\, \epsilon_r\, \sigma^{\mathrm{IT}}(\epsilon_r)\,. \tag{8}$$

It can be used to compute the pion injection for *arbitrary* input proton and photon spectra.

c) Let us now discuss further simplifcations and some approximative interaction types for the Δ-resonance.

1. Re-write Eq. (7) as a single integral by eliminating the δ-function in Eq. (8). Replace the second integral by an integral over y.
2. Compute f^{Δ_1} in Eq. (8) for the (lower) Δ-resonance approximation corresponding to the interaction type "LR" (lower resonance) in Fig. 1. Plot the function: where does it actually peak?
3. Compute now a different approximation f^{Δ_2} for this resonance: Approximate by a δ-function in ϵ_r peaking at 350 MeV. Where does it peak? What is the effect of the width of the resonance by comparison to 2?

Hint. The result of 1) is given by

$$Q_\pi^{\mathrm{IT}}(E_\pi) = N_p \left(\frac{E_\pi}{\chi^{\mathrm{IT}}}\right) \frac{m_p}{E_\pi} \int\limits_{\epsilon_{\mathrm{th}}/2}^{\infty} dy\, n_\gamma \left(\frac{m_p\, y\, \chi^{\mathrm{IT}}}{E_\pi}\right) M_\pi^{\mathrm{IT}} f^{\mathrm{IT}}(y)\,. \tag{9}$$

This (single) integral is relatively simple and fast to compute if $f^{\mathrm{IT}}(y)$ together with χ^{IT} is known.

Simplifed interaction rate. Using the simplifcations above, the interaction rate can be written as

$$\Gamma^{\mathrm{IT}}(E_p) = \frac{m_p}{E_p} \int\limits_{\epsilon_{\mathrm{th}}/2}^{\infty} dy\, n_\gamma \left(\frac{y\, m_p}{E_p}\right) f^{\mathrm{IT}}(y)\,. \tag{10}$$

d) Calculate the interaction rate using the result from c-3) for the following target photon spectrum:

$$n_\gamma(\varepsilon) = \begin{cases} N\,\varepsilon^{-\alpha} & \varepsilon \leq \varepsilon_{\gamma,\mathrm{break}} \\ N\,(\varepsilon_{\gamma,\mathrm{break}})^{\beta-\alpha}\,\varepsilon^{-\beta} & \varepsilon > \varepsilon_{\gamma,\mathrm{break}} \end{cases}, \tag{11}$$

where N is a normalization constant. How does the interaction rate scale as a function of E_p? Use $\alpha = 1$, $\beta = 2$, and $\varepsilon_{\gamma,\mathrm{break}} \sim 1\,\mathrm{keV}$ for specific estimates (GRB-like).

Problem 2: Flavor mixing

Flavor mixing of neutrinos is described as a function of three mixing angles and one phase by the mixing matrix in the standard parameterization

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{12}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$. In the incoherent limit, frequently used in astrophysical applications, the flavor transition probability is given by

$$P_{\nu_\alpha \to \nu_\beta} \equiv P_{\alpha\beta} = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2\,. \tag{13}$$

For the following, assume that astrophysical sources produce ν_e and ν_μ only.

a) Assume that $\theta_{23} \simeq \pi/4$ and $\theta_{13} \simeq 0$, as implied by the "tri-bimaximal" form of the mixing matrix suggested by some flavor symmetries. Derive the flavor transition probabilities from ν_e and ν_μ into all flavors. Check the result by unitarity conservation, for instance, $P_{\mu e} + P_{\mu\mu} + P_{\mu\tau} = 1$.

b) The flux at the detector after flavor mixing ϕ_β can be obtained from the flux before flavor mixing $\hat{\phi}_\alpha$ by

$$\phi_\beta = \sum_{\alpha=e,\mu} P_{\alpha\beta}\, \hat{\phi}_\alpha \, . \tag{14}$$

In the limit of a), what is the ratio between ϕ_μ and ϕ_τ? Does that depend on the initial flavor composition?

c) Suppose that $\hat{\phi}_e : \hat{\phi}_\mu : \hat{\phi}_\tau = 1 : 2 : 0$, was suggested by the pion decay chain. Compute $\phi_e : \phi_\mu : \phi_\tau$ for the limit in a).

d) As suggested by recent T2K results, $\theta_{13} > 0$ (2.5σ). What changes qualitatively? Can one measure the phase δ?